Effective theories of scattering with an attractive inverse-square potential and the three-body problem

2005 J. Phys. A: Math. Gen. 38 697

(http://iopscience.iop.org/0305-4470/38/3/009)

Institute of Physics Publishing | Journal of Physics A: Mathematical and General

J. Phys. A: Math. Gen. **38** (2005) 697–719 doi:10.1088/0305-4470/38/3/009

Effective theories of scattering with an attractive inverse-square potential and the three-body problem

Thomas Barford and Michael C Birse

Theoretical Physics Group, Department of Physics and Astronomy, University of Manchester, Manchester, M13 9PL, UK

Received 9 June 2004, in final form 16 November 2004
Published 23 December 2004
Online at stacks.iop.org/JPhysA/38/697

Abstract

A distorted-wave version of the renormalization group is applied to scattering by an inverse-square potential and to three-body systems. In attractive three-body systems, the short-distance wavefunction satisfies a Schrödinger equation with an attractive inverse-square potential, as shown by Efimov. The resulting oscillatory behaviour controls the renormalization of the three-body interactions, with the renormalization-group flow tending to a limit cycle as the cut-off is lowered. The approach used here leads to single-valued potentials with discontinuities as the bound states are cut off. The perturbations around the cycle start with a marginal term whose effect is simply to change the phase of the short-distance oscillations, or the self-adjoint extension of the singular Hamiltonian. The full power counting in terms of the energy and two-body scattering length is constructed for short-range three-body forces.

PACS numbers: 03.65.Nk, 21.45.+v, 11.10.Hi, 21.30.Fe

1. Introduction

The successful application of effective field theories (EFTs) to two-body scattering has revived interest in developing model-independent treatments of few-body systems, as reviewed in [1, 2]. In the case of two-body systems, EFTs provide a framework within which the old idea of the effective-range expansion [3, 4] can be extended systematically to describe electromagnetic or weak couplings. Their application to three-body systems requires the addition of three-body forces. The observation that such forces are essential for describing low-energy three-body observables goes back to the work of Phillips [5] (see also [6]). One model-independent way of introducing them is the boundary-condition method developed by Brayshaw [7]. The EFT approach provides a more practical way of doing so, and one which can be extended to include couplings to external currents.

For two-body scattering, these EFTs have been well explored. If there is a clear separation between the low-energy scales of interest and the high-energy scales characterizing the underlying physics, then the interactions terms in the effective Lagrangian can be organized systematically as an expansion in powers of ratios of low-energy to high-energy scales. For weakly interacting systems this 'Weinberg' power counting is just that of naive dimensional analysis [8, 9].

In contrast, in strongly interacting systems with shallow resonances or bound states, simple dimensional analysis is no longer appropriate because of the appearance of new low-energy scales. These result in a need to resum certain terms which are 'unnatural' in size [9, 10]. This has been done within various frameworks [10, 11, 13, 14], all of which lead to the same power counting, often referred to as 'KSW counting'. For systems with short-range forces only, it is in fact equivalent to the effective-range expansion developed by Bethe and others [3, 4]. This expansion is the relevant one for few-nucleon systems at low energies, and it may also apply to atomic systems where Feshbach resonances can be tuned to give large scattering lengths.

Both of these power-counting schemes can be understood in terms of fixed points of the Wilsonian renormalization group (RG) [12]. The RG flow near these points defines the power counting for perturbations around them. Weinberg counting arises from the expansion around the trivial fixed point of the RG. KSW counting is associated with a nontrivial fixed point corresponding to a scale-free system with a bound-state exactly at threshold.

The RG approach can also be extended to systems with known long-range interactions, provided one has identified all the low-energy scales. In [15] we applied this to examples including Coulomb and repulsive inverse-square potentials, for which well-defined distorted waves (DWs) exist. In this distorted-wave RG (DWRG), a cut-off is applied to the basis of DWs of the long-range potential, leaving that potential unaffected by the cut-off. The advantage of the method is that it provides a clean separation between the short- and long-range physics. A resulting nontrivial fixed point corresponds to a DW or 'modified' version of the effective-range expansion [3, 16].

Extensions of the EFT description to systems of three particles involve three-body forces. These can be represented by six-point interactions in the Lagrangian or, equivalently, by short-distance boundary conditions ('pseudopotentials', as discussed for example in [10]). In scattering of a third particle from a bound pair, there can be long-range forces resulting from the exchange of one of the particles. The range of these forces is controlled by the two-body scattering lengths. In weakly interacting systems, where the size of the scattering lengths is 'natural', these forces can be treated as short-range, and Weinberg counting applies to the three-body interactions. However, for systems with large two-body scattering lengths, the determination of the power counting is complicated by the presence of long-range forces. Here we apply the DWRG to determining the power-counting for three-body forces. A very brief account of these ideas was previously presented in [17], and a much more extensive one can be found in the PhD thesis [18].

Bedaque *et al* [19, 20] were the first to look at the problem of short-range three-body forces in EFTs. Their method is based on the equation originally derived by Skorniakov and Ter-Martirosian (STM) [24] for systems with zero-range two-body forces. In higher partial waves and in s-wave nuclear systems with spin or isospin $3/2$, centrifugal forces or the Pauli exclusion principle act to keep the particles apart and Bedaque *et al* found that the three-body interactions are irrelevant (in the technical RG sense that they vanish as the cut-off is lowered). In contrast, for three bosons in s-waves or three s-wave nucleons with spin and isospin $1/2$, Bedaque *et al* found that three-body forces are not merely important in these systems, they are in fact essential for producing well-defined results [19, 20] (see also [21, 22]). This work provides a systematic framework for understanding the 'Phillips line', correlating low-energy three-body observables [5, 6]. More recently, a power-counting scheme for the three-body forces in these systems has been constructed by renormalizing the STM equation order-by-order in the energy [23]. Our work confirms this power counting within the framework of a full RG analysis.

In other recent work on this problem, Phillips and Afnan [25] have looked at the STM equation in the case of three s-wave bosons. They were able to reproduce the results of Bedaque *et al* by using a subtractive renormalization and introducing a single piece of three-body data, namely the three-body scattering length. This subtraction constant is equivalent to the leading three-body force of [20]. Similar results have also been obtained by Mohr [26].

An alternative to the STM equation for three-body systems with contact interations is provided by the work of Efimov [27] (for a review, see [28]). This showed that, in the case of infinite two-body scattering length, the equations become separable in hyperspherical coordinates and the resulting Hamiltonian contains an inverse-square potential (ISP). The strength of this potential is determined by the statistics of the system. In systems with nonzero orbital angular momentum or where the Pauli exclusion principle keeps the particles apart, it is repulsive. However, in systems with no angular momentum or exclusion principle, such as three s-wave bosons, there can be an attractive ISP. These are the cases where a three-body force is required.

Because of its relevance to three-body EFTs, the attractive ISP has been the subject of several recent papers [29–31]. It will also play a central role in this paper, since the DWRG analysis for scattering in the presence of this potential provides the basic power counting for the three-body system. The attractive ISP is particularly interesting from a mathematical point of view because its wavefunctions have a logarithmic oscillatory behaviour near the origin [32], which makes it impossible to define a 'regular' wavefunction at the origin. Mathematically, the resolution of this quandary is well known: we need to form a self-adjoint extension of the Hamiltonian [33–35]. More physically, one can think of this as introducing a boundary condition to fix the phase of the oscillatory solutions near the origin.

An alternative way to fix the self-adjoint extension is to regulate the singularity of the potential at short distances and to introduce a counterterm [29, 36]. In essence this is what momentum-space regularization of the STM equation does for the three-body problem [20, 22, 23]. Other authors have shown explicitly how a short-range force can be used to select an extension [29, 31, 36], but have not developed the power counting for it. Furthermore, the three-body forces obtained in this manner are multi-valued, with no obvious procedure to resolve this [29, 31].

A notable feature of the approaches based on regularization of the singular interaction is that they exhibit a cyclic behaviour as the scale of the cut-off is varied [20, 22, 29, 36]. In fact, as pointed out by Wilson [37], they form a novel kind of limit cycle of the RG. Several examples of this have only recently been highlighted [38–40]. In addition, Braaten and Hammer have pointed out that a rather minor fine tuning of the quark masses in QCD would bring the infrared limit of the theory to the effective-range fixed point in the two-nucleon channel, and hence to a limit cycle for three nucleons [41].

In this paper we apply the DWRG method, first to the attractive ISP corresponding to three-body systems with infinite two-body scattering length, and then to more general attractive three-body systems. We apply a cut-off to the deeply bound states as well as the continuum, and this leads to an RG equation with single-valued but discontinuous solutions. By taking advantage of Efimov's separation of the full three-body wavefunction, we are able to show that the power counting of [23] is not limited to the STM 'slice' through it, where two of the particles coincide. Because of the clean separation of short- and long-range physics in the DWRG, we are able to derive this counting in a more transparent way. By constructing the physical scattering amplitudes, we are able to directly relate the coefficients in the three-body potential to observables, through a version of the DW effective-range expansion.

2. The RG for three-body forces

For simplicity, we consider here a system of three bosons of mass M, in a state of zero orbital angular momentum. We assume that the two-body interaction produces a single shallow bound state with binding energy γ^2/M, corresponding to a 'binding momentum' $\mathrm{i}\gamma$. We start by considering the effects of two-body forces only and we denote the corresponding three-body Green function by $G_2(p)$, where p is defined in terms of the total centre-of-mass energy by $E = p^2/M$. This function has the spectral decomposition,

$$G_2(p) = \frac{M}{4\pi}\sum_{n=1}^{N}\frac{|\Psi_n\rangle\langle\Psi_n|}{p^2+p_n^2} + \frac{M}{2\pi^2}\int_{-\gamma^2}^{\infty}\mathrm{d}(q^2)\frac{1}{p^2-q^2+\mathrm{i}\epsilon} \times\left[|\Psi_{q,\mathrm{i}\gamma}\rangle\langle\Psi_{q,\mathrm{i}\gamma}| + \vartheta(q^2)\frac{2}{\pi}\int_0^q \mathrm{d}k|\Psi_{q,k}\rangle\langle\Psi_{q,k}|\right], \tag{1}$$

where $\vartheta(x)$ is the unit step function. At this point, we should note that an attractive three-body system with two-body contact interactions is not mathematically well-defined, and an additional boundary condition is needed to specify the DWs uniquely. As discussed below, the effects of this boundary condition are equivalent to a three-body force and so, strictly speaking, it is not possible to define such a system with only two-body forces.

Three types of waves appear in this decomposition. In the first term, we have the bound states of three particles, $|\Psi_n\rangle$. In the final term, we have states with three incoming free particles, $|\Psi_{q,k}\rangle$, which we label by their total centre-of-mass energy, $E = q^2/M$, and relative momentum, k. In the middle are states with one incoming free particle and a bound pair $|\Psi_{q,\mathrm{i}\gamma}\rangle$. These states are normalized as described in appendix A.

Our short-distance effective three-body interactions act in the in the region where all three particles are close together. In coordinate space, a natural measure of the proximity of the three particles is the hyperradius, R, defined in appendix A. We write the DWs as functions of R and the five hyperspherical angles, collectively denoted by Ω, which represent the other degrees of freedom in the centre-of-mass frame. The precise specification of these angles is not needed since all Ω dependence will factor out of our results.

A three-body interaction, V_3, could be inserted directly into the Faddeev equations [42]. However, for our purposes, it is sufficient to use the 'two-potential trick' [32] to define a T-matrix for the additional scattering produced by V_3. We write the full $3 \to 3$ T-matrix, $T(p)$, in terms of the one with two-body forces only, $T_2(p)$, plus an additional piece, $\tilde{T}_3(p)$, that acts between the DWs for the two-body forces:

$$\langle q,k|T(p)|q',k'\rangle = \langle q,k|T_2(p)|q',k'\rangle + \langle\Psi_{q,k}|\tilde{T}_3(p)|\Psi_{q',k'}\rangle. \tag{2}$$

The DW term $\tilde{T}_3(p)$ satisfies the Lippmann–Schwinger (LS) equation,

$$\tilde{T}_3(p) = V_3 + V_3 G_2(p)\tilde{T}_3(p). \tag{3}$$

There is no problem with connectedness in this equation since the kernel contains the three-body force.

For zero-range two-body interactions, the small-R behaviour of the wavefunctions in equation (1) can be found from the work of Efimov [27]. The essential elements of this, in our notation, are outlined in appendix A. Since the wavefunctions do not have well-defined limits as $R \to 0$, we choose our effective three-body interaction to act at some small, but nonzero hyperradius, $R = \bar{R}$. This additional regularization is quite separate from the running cut-off which will give the RG flow. As with the singular long-range potentials studied in [15], it is required to make all wavefunctions appearing in the RG equation well defined. Provided we choose $\bar{R}$ to be small enough that the waves have reached their asymptotic forms, the common dependence on R can be factored. In addition, we take the interaction to act at some fixed set of hyperangles $\Omega = \bar{\Omega}$. Since the wavefunctions at small R are separable, all hyperangular behaviour can also be factored out and this arbitrary choice will not affect the results. We can, therefore, define

$$\langle \Psi_{p,k} | V_3 | \Psi_{p,k'} \rangle = \bar{R}^4 \Psi^*_{p,k}(\bar{R}, \bar{\Omega}) \Psi_{p,k'}(\bar{R}, \bar{\Omega}) V_3(p, k, k'). \tag{4}$$

In an EFT we aim to integrate out all the unknown short-range physics and replace it by a three-body force which can be expanded in powers of the low-energy scales of the system. The tool which allows us to determine the scaling behaviour of the terms in V_3 is the RG [12]. The first step in setting this up is to impose a cut-off, Λ, to separate the low-energy states which we wish to treat explicitly from the high-energy states which will be integrated out.

As in [15], we impose this cut-off on the DWs in equation (1). This ensures that the role of the three-body force is related purely to short-range three-body physics. If the cut-off were applied to free waves, it would also remove parts of the physics associated with the two-body forces. This would have to be compensated by the three-body force, which would then satisfy a far more complicated evolution equation.

We then renormalize the theory by making the effective interactions Λ-dependent and demanding that observables should be independent of the cut-off. In particular we require that $\tilde{T}_3$ be independent of cut-off, $\partial_\Lambda \tilde{T}_3 = 0$. Differentiating the LS equation (3) for $\tilde{T}_3$ then leads, after the elimination of $\tilde{T}_3$, to a differential equation for V_3,

$$\frac{\partial}{\partial \Lambda} V_3(\Lambda) = -V_3(\Lambda) \frac{\partial G_2(p, \Lambda)}{\partial \Lambda} V_3(\Lambda). \tag{5}$$

The RG equation is then obtained by rescaling the potential V_3 and all the low-energy scales in this equation. As described below, the definition of the rescaled potential $\hat{V}_3$ depends upon the form of the DWs close to the origin. The boundary conditions on $\hat{V}_3$ are that it should be analytic in all rescaled energies and momenta. These follow from our requirement that the terms in the three-body force arise from local six-point vertex terms in the EFT Lagrangian. The Λ-dependence of the corresponding terms in $\hat{V}_3$ then allows us to classify them according to some power counting.

The two-body forces in our EFT can be represented by contact interactions, or equivalently by a boundary condition on the logarithmic derivative of the wavefunction when two of the particles coincide [10]. This boundary condition, combined with the overall symmetries of the wavefunction, leads to Efimov's boundary condition on the three-body wavefunction [27], as outlined in appendix A. This becomes separable in the limit where the hyperradius, R, is much smaller than the two-body scattering length, a_2.

After separating the Schrödinger equation, the long-range potential in each hyperangular-momentum channel is an ISP (as it must, since all scales have been eliminated from the problem). For three bosons in s-waves or two neutrons and a proton with total spin $1/2$, the ISP in the lowest hyperangular momentum is attractive. In order to resolve the ambiguity in the wavefunctions for this potential, we need to form a self-adjoint extension.

We can do this by introducing a scale p_* to fix the phase of the waves as $R \to 0$. As shown in appendix A, the form of the resulting DWs at small R is

$$\Psi(R, \Omega) \sim \xi_{s_0}(\Omega) \sin(s_0 \ln(p_* R) - \theta), \tag{6}$$

where $s_0 \simeq 1.006$ is the magnitude of the smallest hyperangular momentum, θ is defined in equation (A.14) and $\xi_{s_0}(\Omega)$ is the hyperangular wavefunction. This form is unchanged (up to an overall sign) by the replacement,

$$p_* \rightarrow p_* \mathrm{e}^{n\pi/s_0}, \qquad n \in \mathbb{Z}, \tag{7}$$

and so all of the values of p_* in equation (7) define the same self-adjoint extension.

3. Infinite scattering length

In the limit of infinite two-body scattering length, Efimov's separation of the Hamiltonian is exact. If we assume also that the three-body force is diagonal in the hyperangular momentum then the RG equations for the various channels are uncoupled. We introduce this approximation here so that we can focus on the lowest hyperangular-momentum channel, which contains the leading interactions. Later, when we derive general results, we shall not require it.

The higher hyperangular-momentum channels, s_i with $i > 0$, correspond to repulsive ISPs. Their RG analysis is thus identical to the one in [15]. The LO terms behave like $(p/\Lambda_0)^{2s_i}$, where Λ_0 is the scale of the underlying physics. The most important of these is the LO interaction in the $s_1 = 4$ channel, but even this is heavily suppressed. Similarly, in systems which do not lead to attractive ISPs, the three-body forces are always suppressed by powers of p/Λ_0. Here we concentrate on the s_0 channel in attractive systems, where three-body forces are enhanced.

3.1. RG equation

The hyperradial Green function in the s_0 channel is given by

$$g_2(p; R, R') = \frac{M}{2\pi} \sum_{n=-\infty}^{\infty} \frac{u_{s_0}^{(n)}(R) u_{s_0}^{(n)}(R')}{p^2 + p_n^2} + \frac{M}{\pi^2} \int_0^{\infty} \mathrm{d}q \, \frac{u_{s_0}(q, R) u_{s_0}(q, R')}{p^2 - q^2 + \mathrm{i}\epsilon}. \tag{8}$$

We use lower case to denote quantities in this channel, for example this Green function and the potential v_3, to differentiate them from those for the full problem. The DWs $u_{s_0}(p, R)$ are simply solutions of the ISP Schrödinger equation (A.9). The bound states $u_{s_0}^{(n)}(R)$ have energies which, from equation (A.17), form an infinite tower with geometric spacing,

$$E_n = -\frac{p_*^2}{M} \mathrm{e}^{2n\pi/s_0}. \tag{9}$$

These respect the symmetry of equation (7).

Before we can write down the DWRG equation for the three-body force in this channel, we must first decide how to handle the bound states. In the previous examples studied with the DWRG method [15], the cut-off was applied purely to the high-energy continuum states. In those examples, any bound states were shallow, with typical momenta much smaller than the scale of the underlying physics, and so lay within the domain of the EFT. Here, however, the lack of a ground state means that this is no longer true. The deeply bound states are unphysical artefacts of our use of contact interactions to represent the two-body forces. In keeping with the EFT philosophy these states should be truncated and their effects absorbed into the effective three-body force. We choose to cut-off the Green function by removing all states with energies outside the range $-\Lambda^2/M \leqslant E \leqslant \Lambda^2/M$:

$$g_2(p; R, R') = \frac{M}{2\pi} \sum_{|p_n|<\Lambda} \frac{u_{s_0}^{(n)}(R) u_{s_0}^{(n)}(R')}{p^2 + p_n^2} + \frac{M}{\pi^2} \int_0^{\Lambda} \mathrm{d}q \, \frac{u_{s_0}(q, R) u_{s_0}(q, R')}{p^2 - q^2 + \mathrm{i}\epsilon}. \tag{10}$$

We shall see that this truncation will lead to an RG equation with single-valued solutions.

Using the separable form of the wavefunctions in equation (6), we can project the differential equation (5) onto the s_0 channel. We can then absorb the common angle-dependent factor $|\xi_{s_0}(\bar{\Omega})|^2$ into into the potential by defining

$$v_3(p,\Lambda) = |\xi_{s_0}(\bar{\Omega})|^2 V_3(p,\Lambda). \tag{11}$$

Note that we have chosen to focus on potentials that depend only on energy since, as discussed in [13, 15], these contain the leading perturbations. Inserting equation (10) into the projected equation we get

$$\frac{\partial v_3(p,\Lambda)}{\partial\Lambda} = \frac{M}{\pi^2}\left[\frac{|u_{s_0}(\Lambda,\bar{R})|^2}{\Lambda^2 - p^2} - \frac{\pi}{2}\sum_{n=-\infty}^{\infty}\frac{\left|u_{s_0}^{(n)}(\bar{R})\right|^2}{p^2+p_n^2}\delta(\Lambda - p_n)\right] v_3(p,\Lambda)^2. \tag{12}$$

The first term is produced by cutting off the continuum states, and is similar to the expressions in [13, 15]. The series of discontinuities represented by δ-functions results from the truncation of the bound states.

Finally, to obtain an RG equation we need to rescale any low-energy scales in the problem. In this case there is only on-shell momentum p, and we define $\hat{p} = p/\Lambda$. To form a dimensionless potential, we multiply it by the mass M. We also take the expressions for the wavefunctions near the origin, equations (A.13, A.19), and absorb the $\bar{R}$-dependence into the potential by defining

$$\hat{v}_3(\hat{p},\Lambda) = \frac{M}{s_0\pi^2}\sin^2(s_0\ln(p_*\bar{R}) - \theta)v_3(\hat{p}\Lambda,\Lambda). \tag{13}$$

The rescaled potential $\hat{v}_3$ then satisfies the RG equation

$$\Lambda\frac{\partial\hat{v}_3}{\partial\Lambda} = \hat{p}\frac{\partial\hat{v}_3}{\partial\hat{p}} + \left[\frac{\sinh(\pi s_0)}{[\cosh(\pi s_0) - \cos(2s_0\ln(\Lambda/p_*))](1-\hat{p}^2)} - \frac{\pi}{s_0}\sum_{n=-\infty}^{\infty}\frac{1}{1+\hat{p}^2}\delta(\hat{p}_n(\Lambda) - 1)\right]\hat{v}_3^2, \tag{14}$$

where $\hat{p}_n(\Lambda) = p_n/\Lambda = p_*\,\mathrm{e}^{n\pi/s_0}/\Lambda$. Note that we write $\hat{p}_n(\Lambda)$ as a function of Λ since we do not rescale p_*, and so $\hat{p}_n$ varies with Λ. As noted before [15], this type of RG equation is more conveniently rewritten as a linear equation for $1/\hat{v}_3$,

$$\Lambda\frac{\partial}{\partial\Lambda}\left(\frac{1}{\hat{v}_3}\right) = \hat{p}\frac{\partial}{\partial\hat{p}}\left(\frac{1}{\hat{v}_3}\right) - \frac{\sinh(\pi s_0)}{[\cosh(\pi s_0) - \cos(2s_0\ln(\Lambda/p_*))](1-\hat{p}^2)} + \frac{\pi}{s_0}\sum_{n=-\infty}^{\infty}\frac{1}{1+\hat{p}^2}\delta(\hat{p}_n(\Lambda) - 1). \tag{15}$$

This RG equation is the same as that for an attractive ISP in two dimensions. In fact an attractive ISP in any number of dimensions leads to an equation of this form, because the different real power of $\bar{R}$ which appears in the short-distance wavefunctions exactly compensates for the radial factor in the Jacobian. All that differs is a numerical factor in the definition of $\hat{v}_3$, associated with the angular integration.

3.2. Solutions

The physically acceptable solutions to equation (14) must be analytic in $\hat{p}^2$ for small $\hat{p}$. We look first for fixed points, Λ-independent solutions. The power counting for terms in the potential can be determined from the perturbations around a fixed point that scale with definite powers of Λ [13, 15].

An obvious fixed point is the trivial one, $\hat{v}_3 = 0$. The power counting based on it can be found by substituting

$$\hat{v}_3 = C\Lambda^{\mu}\phi(\hat{p}) \tag{16}$$

into the RG equation (14), and linearizing to get the eigenvalue equation

$$\hat{p}\frac{d\phi}{d\hat{p}} = \mu\phi. \tag{17}$$

This is easily solved, giving $\phi(\hat{p}) = \hat{p}^{\mu}$. Imposing the boundary condition of analyticity in $\hat{p}^2$, leads to the eigenvalues $\mu = 0, 2, 4, \ldots$. Hence the general solution in the region of this fixed point is

$$\hat{v}_3(\Lambda, \hat{p}) = \sum_{n=0}^{\infty} C_{2n}\Lambda^{2n}\hat{p}^{2n}. \tag{18}$$

The leading term in this expansion is marginal, that is, it does not scale with any power of Λ. We therefore expect to find logarithmic dependence on Λ associated with this perturbation.

These logarithms become large as $\Lambda \to 0$ and to resum them we need to construct solutions of the full nonlinear RG equation. In fact Λ-dependence of the right-hand side of equation (14) means that no other fixed-point solution can be found. However, that dependence is only logarithmic and so we may look for a solution that also depends only logarithmically on Λ. Perturbations about such a solution can still be used to construct a power counting.

As in [15], our starting point for constructing a nontrivial solution to the RG is the basic loop integral, which in this case is given by

$$\hat{I}(\hat{p}, \Lambda) = \mathcal{P}\int_0^1 d\hat{q}\frac{\hat{q}\sinh(\pi s_0)}{[\cosh(\pi s_0) - \cos(2s_0\ln(\Lambda\hat{q}/p_*))](\hat{p}^2 - \hat{q}^2)}. \tag{19}$$

By substituting this for $1/\hat{v}_3$ in equation (15), it is straightforward to verify that it satisfies the continuous version of the equation (without the discontinuities of the final term). However, it is not yet an acceptable solution since, apart from missing the bound state discontinuities, it is nonanalyitic for small $\hat{p}$.

The propagator pole at $\hat{q} = \hat{p}$ in the integrand approaches the endpoint of the integral as $\hat{p} \to 0$, leading to logarithmic dependence of $\hat{I}$ on $\hat{p}$. To avoid this, we look for a similar integral over some contour in the complex $\hat{q}$-plane that avoids the singular region.

We first rewrite the integrand as

$$\frac{1}{2i}\frac{\hat{q}}{(\hat{p}^2 - \hat{q}^2)}\left[\cot\left(s_0\ln\frac{\Lambda\hat{q}}{p_*} - \frac{i\pi s_0}{2}\right) - \cot\left(s_0\ln\frac{\Lambda\hat{q}}{p_*} + \frac{i\pi s_0}{2}\right)\right], \tag{20}$$

so that I becomes

$$I(\hat{p}, \Lambda) = \mathcal{P}\int_{-1}^{1} d\hat{q}\, h(\hat{q}), \tag{21}$$

where

$$h(\hat{q}) = \frac{1}{2i}\frac{\hat{q}}{(\hat{p}^2 - \hat{q}^2)}\cot\left(s_0\ln\frac{\Lambda\hat{q}}{p_*} - \frac{i\pi s_0}{2}\right). \tag{22}$$

The dangerous region around $\hat{q} = 0$ is not at an endpoint of this integral, and so we are free to avoid it by deforming the contour of integration into the complex plane. We therefore define

$$\hat{J}(\hat{p}, \Lambda) = \int_C d\hat{q}\, h(\hat{q}), \tag{23}$$

where the contour C runs from $\hat{q} = -1$ to $\hat{q} = 1$ in the upper half plane, as in figure 1.

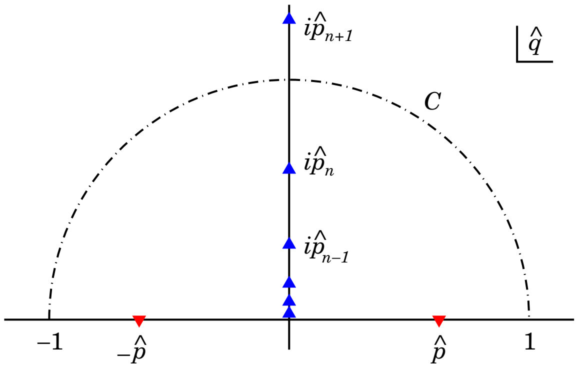

Figure 1. The contour C in the complex $\hat{q}$-plane used to construct the solution $\hat{\jmath}(\hat{p}, \Lambda)$ of the DWRG equation. The bound-state poles of the integrand occur at $\hat{q} = \mathrm{i}\hat{p}_n = \mathrm{i}p_* \mathrm{e}^{n\pi/s_0}/\Lambda$ and the propagator poles at $\hat{q} = \pm\hat{p}$.

Figure 1 also shows the pole structure of the integrand $h(\hat{q})$. Apart from the two propagator poles at $\hat{q} = \pm\hat{p}$, there are bound-state poles at

$$\hat{q} = \mathrm{i}\hat{p}_n(\Lambda) = \mathrm{i}\frac{p_*}{\Lambda}\,\mathrm{e}^{n\pi/s_0}. \tag{24}$$

These poles are important since their positions vary with Λ and so they can cross our integration contour. When this happens they generate discontinuities in $\hat{\jmath}(\hat{p}, \Lambda)$. By choosing our contour to cross the imaginary axis at $\hat{q} = \mathrm{i}$ (corresponding to an energy $E = -\Lambda^2/M$), we can arrange for these discontinuities to match precisely those we require in equation (15). To see this, note that as Λ is varied from $p_n - \epsilon$ to $p_n + \epsilon$, the pole at $\hat{q} = \mathrm{i}\hat{p}_n$ crosses the contour and produces a discontinuity

$$[\hat{\jmath}(\hat{p}, \Lambda)]_{\Lambda=p_n-\epsilon}^{\Lambda=p_n+\epsilon} = -2\pi\mathrm{i}\mathcal{R}[h(\hat{q}), \mathrm{i}\hat{p}_n] = \frac{\pi}{s_0}\frac{1}{1+\hat{p}^2}, \tag{25}$$

where $\mathcal{R}[f(z), z_0]$ denotes the residue of $f(z)$ at the pole z_0. This is precisely the strength of the δ-function at $p_n(\Lambda) = 1$ in equation (15).

The integral $\hat{\jmath}(\hat{p}, \Lambda)$ satisfies the full RG equation (15) and is analytic about $\hat{p}^2 = 0$. We therefore define the rescaled potential by

$$\hat{v}_3^{(0)}(\hat{p}, \Lambda) = [\hat{\jmath}(\hat{p}, \Lambda)]^{-1}. \tag{26}$$

The logarithmic dependence on Λ has been resummed to all orders in this potential. The invariance under equation (7) means that the dependence on $\ln\Lambda$ is periodic. As pointed out by Wilson [37], this periodic behaviour provides an example of a limit cycle of the RG. The cycle contains one discontinuity every period, when a bound state is removed from the low-energy domain.

Like a fixed point, the limit-cycle solution can be used to define a power counting for the perturbations around it. Because the RG equation in the form of equation (15) is linear in $1/\hat{v}_3$, it is possible to construct exact solutions containing these perturbations,

$$\frac{1}{\hat{v}_3(\hat{p}, \Lambda)} = \frac{1}{\hat{v}_3^{(0)}(\hat{p}, \Lambda)} + \sum_{n=0}^{\infty} C_{2n}\Lambda^{2n}\hat{p}^{2n}. \tag{27}$$

The perturbations around the cycle thus have exactly the same power counting as those around the trivial fixed point. The leading, energy-independent term is marginal. Each additional power of energy ($\hat{p}^2$) gives a term two orders higher in Λ.

The general solution, equation (27), shows that there is a family of limit cycles parametrized by the marginal perturbation C_0. This lack of uniqueness arises because the marginal perturbation is Λ-independent and satisfies the homogeneous version of the RG equation (15). As a result, an arbitrary amount of it can be added to the limit-cycle solution, equation (26). The leading perturbation around any cycle is marginal because it pushes the solution into a nearby cycle. In the limit $C_0 \to \infty$, the entire cycle is compressed into the trivial fixed point $\hat{v}_3 = 0$, which can be regarded as the limiting member of the family of cycles. All other perturbations vanish as $\Lambda \to 0$ and hence are stable. Any more general solution will thus tend to one of the limit cycles as $\Lambda \to 0$.

The truncation of bound states ensures that our solution of the RG equation is single valued. Our choice of a cut-off that is symmetric between positive and negative energies corresponds to a particular path from $\hat{q} = -1$ to +1 for the contour C. Other prescriptions are equally acceptable since physical observables should not depend on them. The resulting solutions, corresponding to different choices of path, could be regarded as different branches of a general multi-valued solution. Such multi-valuedness appears in other methods used to renormalize the three-body problem, as in [20–23].

3.3. Scattering observables

To relate the parameters appearing in our solution, equation (27), to scattering observables, we need to construct the corresponding T-matrix. If we use the separable form of the wavefunctions, we can define a DW T-matrix $\tilde{t}_3$ in an analogous manner to equation (12). Projecting the LS equation (3) onto the s_0 channel, we get the corresponding equation for $\tilde{t}_3$. With our choice of energy-dependent potential, this can be solved directly to get the on-shell T-matrix,

$$\langle u_p^- | \tilde{t}_3(p) | u_p^+ \rangle = \mathrm{e}^{2\mathrm{i}\delta_2(p)} |u_p(\bar{R})|^2 v_3(p, \Lambda)[1 - g_2(p, \Lambda; \bar{R}, \bar{R}) v_3(p, \Lambda)]^{-1}, \tag{28}$$

where the superscripts $\pm$ denote waves with incoming or outgoing boundary conditions, and δ_2 is the phase shift in the s_0 channel produced by the pairwise forces alone.

Unitarity and conservation of hyperangular momentum allow us to express this in terms of a phase-shift as

$$\frac{\mathrm{e}^{2\mathrm{i}\delta_2(p)}}{\langle u_p^- | \tilde{t}_3(p) | u_p^+ \rangle} = -\frac{M}{2\pi p}(\cot \tilde{\delta}_3(p) - \mathrm{i}). \tag{29}$$

Here $\tilde{\delta}_3$ is the additional phase shift produced by the three-body force. It is related to the full phase shift δ by $\tilde{\delta}_3 = \delta - \delta_2$. Using the fact that our potential acts at $R = \bar{R}$, we can combine equations (28) and (29) to obtain an equation relating v_3 and the phase shift:

$$|u_p(\bar{R})|^2 \frac{M}{2\pi p}(\cot \tilde{\delta}_3 - \mathrm{i}) = g_2(p, \Lambda; \bar{R}, \bar{R}) - v_3(p, \Lambda)^{-1}. \tag{30}$$

When we use the expression in equation (10) for the regularized Green function and the explicit forms of the wavefunctions from appendix A, we find that the $\bar{R}$ dependence can be factored out to leave

$$\begin{aligned} &\frac{\pi}{2}\left(\frac{\sinh(\pi s_0)}{\cosh(\pi s_0) - \cos(2 s_0 \ln(p/p_*))}\right)(\cot \tilde{\delta}_3(p) - \mathrm{i}) \\ &\quad = \int_0^\Lambda \frac{q \sinh(\pi s_0)\, \mathrm{d}q}{[\cosh(\pi s_0) - \cos(2 s_0 \ln(q/p_*))](p^2 - q^2 + \mathrm{i}\epsilon)} \\ &\qquad + \frac{\pi}{s_0} \sum_{|p_n|<\Lambda} \frac{p_n^2}{p^2 + p_n^2} - \frac{1}{\hat{v}_3(p/\Lambda, \Lambda)}. \end{aligned} \tag{31}$$

To write this in a form which is explicitly independent of Λ, we introduce the rescaled variable $\hat{q} = q/\Lambda$ and substitute our solution for $\hat{v}_3$ from equation (27). This gives

$$\frac{\pi}{2}\left(\frac{\sinh(\pi s_0)}{\cosh(\pi s_0) - \cos(2s_0 \ln(p/p_*))}\right)(\cot\tilde{\delta}_3(p) - \mathrm{i}) = \int_{-1}^{1} \mathrm{d}\hat{q}\, h(\hat{q}) - \int_C \mathrm{d}\hat{q}\, h(\hat{q}) - 2\pi\mathrm{i} \sum_{|p_n|<\Lambda} \mathcal{R}[h(\hat{q}), \mathrm{i}p_n/\Lambda] - \sum_{n=0}^{\infty} C_{2n} p^{2n}, \qquad (32)$$

where $h(\hat{q})$ is defined above in equation (21), and we have used the fact that the sum over the bound states can be expressed in terms of the residues at the poles (see equation (25)).

The two integrals over $h(\hat{q})$ can be combined to form a single integral round a closed contour. The iϵ prescription in the first integral implies that the contour of integration encloses the propagator pole at $\hat{q} = \hat{p}$ but not the one at $\hat{q} = -\hat{p}$. It also encloses the bound-state poles at $\hat{q} = \mathrm{i}p_n/\Lambda$ with $p_n < \Lambda$. Using Cauchy's theorem, we find that the residues from these exactly cancel the sum over bound states in equation (32).

Finally, we are left with a DW effective-range expansion,

$$\left(\frac{\sinh(\pi s_0)}{\cosh(\pi s_0) - \cos(2\eta(p))}\right)(\cot\tilde{\delta}_3(p) - \mathrm{i}) = \cot\left(\eta(p) + \frac{\mathrm{i}\pi s_0}{2}\right) - \frac{2}{\pi}\sum_{n=0}^{\infty} C_{2n} p^{2n}, \qquad (33)$$

where $\eta(p)) = -s_0 \ln(p/p_*)$ is defined in equation (A.15). Making use of trigonometric addition formulae, one can show that the imaginary parts of the left and right sides are equal. In this expansion, all nonanalytic behaviour has been subtracted or factored out into the trigonometric functions of $\ln p$. The remaining energy dependence can be expanded in powers of p^2, and the terms correspond directly to perturbations around the limit cycle.

The roles of the limit-cycle potential and its marginal perturbation C_0 are not obvious from equation (33) and so, to clarify these, we construct the total phase shift, δ. Making use of equations (33) and (A.16), the corresponding full S-matrix can be written as

$$\mathrm{e}^{2\mathrm{i}\delta(p)} = \mathrm{i}\frac{Z^*(p)}{Z(p)}, \qquad (34)$$

where

$$Z(p) = \cos\left(\eta(p) + \frac{\mathrm{i}\pi s_0}{2}\right) - \frac{2}{\pi}\sin\left(\eta(p) + \frac{\mathrm{i}\pi s_0}{2}\right)\sum_{n=0}^{\infty} C_{2n} p^{2n}. \qquad (35)$$

For the limit-cycle solution of equation (26) with no perturbations (all $C_{2n} = 0$), we have

$$Z(p) = \cos\left(\eta(p) + \frac{\mathrm{i}\pi s_0}{2}\right). \qquad (36)$$

Comparing this to equation (A.16) for the pure long-range force, we see that the phase $\eta(p)$ of the waves near the origin has been shifted by $\pi/2$. This implies, for example, that the bound states lie at $p = \mathrm{i}\mathrm{e}^{-\pi/2s_0} p_n$, which correspond to the geometric means of the bound state energies for $v_3 = 0$. This potential thus corresponds to the system which is 'furthest away' from the one with $v_3 = 0$, in the sense that it leads to the maximum possible changes to physical observables.

To elucidate the role of the marginal perturbation, it is convenient to express its coefficient in terms of an angle σ,

$$C_0 = -\frac{\pi}{2}\cot\sigma, \qquad (37)$$

and to redefine the other short-distance parameters according to

$$\frac{2}{\pi}\sum_{n=1}^{\infty} C_{2n} p^{2n} = \frac{\csc\sigma \sum_{n=1}^{\infty} C'_{2n} p^{2n}}{\sin\sigma + \cos\sigma \sum_{n=1}^{\infty} C'_{2n} p^{2n}}. \qquad (38)$$

In terms of the new parameters, σ and C'_{2n}, $n \geqslant 1$ we find that the S-matrix can still be written in the form of equation (34), but with $Z(p)$ replaced by

$$Z'(p) = \sin\left(\eta(p) + \sigma + \frac{\mathrm{i}\pi s_0}{2}\right) + \cos\left(\eta(p) + \sigma + \frac{\mathrm{i}\pi s_0}{2}\right) \sum_{n=1}^{\infty} C'_{2n} p^{2n}. \qquad (39)$$

This form makes it clear that σ (or rather C_0) just has the effect of shifting the phase $\eta(p)$. Scattering observables depend only on the combination

$$p'_* = \mathrm{e}^{-\sigma/s_0} p_*, \qquad (40)$$

and not on p_* and C_0 separately. This shows that p_* and C_0 play the same role in determining the phase of the wavefunctions for $R \to 0$ (the self-adjoint extension of the Hamiltonian). We can use C_0 to change the self-adjoint extension from the initial one specified by p_* to any other. In particular, $\sigma = 0$ (corresponding to the trivial fixed point) leaves the initial extension unchanged, whereas $\sigma = \pi/2$ ($C_0 = 0$) produces the largest possible change in the extension, as already noted. Furthermore, there is a one-to-one mapping between all possible limit-cycle solutions (obtained by varying C_0 from $-\infty$ to $+\infty$ or equivalently σ between 0 and π) and the self-adjoint extensions. This is in marked contrast to the relationship between p_* and the self-adjoint extensions, where infinitely many equivalent p_* give the same extension.

The bound states of the system are given by the zeros of $Z'(p)$. For any short-range potential, the bound states still accumulate at zero energy, as expected since they are consequences of the inverse-square tail of the long-range potential. The shallower bound states are insensitive to the short-range perturbations, C'_{2n} with $n \geqslant 1$. Their positions are controlled by p'_* rather than the original p_*. If we choose p_* to give the correct shallow Efimov states, we can set $\sigma = 0$ and use the expansion around the trivial fixed point. This corresponds to expanding a DW K-matrix and has the form

$$\frac{\tan\tilde{\delta}_3(p)}{\sinh(\pi s_0)} = \frac{\sum_{n=1}^{\infty} C'_{2n} p^{2n}}{\cosh(\pi s_0) - \cos(2\eta(p)) + \sin(2\eta(p)) \sum_{n=1}^{\infty} C'_{2n} p^{2n}}. \qquad (41)$$

The deeper bound states are of course strongly affected by the higher order perturbations and will not in general follow the simple constant-ratio pattern of the Efimov states. At some point they will fall outside the range of validity of the EFT, and so we can say nothing about them, except that some short-range physics must act to ensure the existence of a ground state.

4. Finite scattering length

In the general case of finite scattering length it no longer makes sense to expand observables in terms of the hyperangular wavefunctions, instead we must consider a much more general RG equation. In particular, the low-energy scales now include $\gamma = 1/a_2$ where a_2 is the two body scattering length. This can also be thought of as the typical momentum in the low-lying bound or virtual state of two particles. The coupling between waves with different hyperangular momenta means that the states should now depend on the relative momentum k of a pair, as well as the total energy, expressed as a momentum p. Nonetheless, Efimov's separation of variables still applies whenever the distances between the three particles are all much less than the two-body scattering length [27]. It, therefore, controls the short-distance behaviour of the three-body wavefunction in an EFT with contact interactions and hence the scaling of the short-range three-body interactions. For attractive three-body systems, the the basic pattern of the RG evolution is similar to that in the previous.

The truncated Green function is given by equation (1) restricted to states inside the energy range $-\Lambda^2/M \leqslant E \leqslant \Lambda^2/M$:

$$G_2(p) = \frac{M}{4\pi}\sum_{|p_n|<\Lambda}\frac{|\Psi_n\rangle\langle\Psi_n|}{p^2+p_n^2} + \frac{M}{2\pi^2}\int_{-\gamma^2}^{\Lambda^2}\mathrm{d}(q^2)\frac{1}{p^2-q^2+\mathrm{i}\epsilon} \times\left[|\Psi_{q,\mathrm{i}\gamma}\rangle\langle\Psi_{q,\mathrm{i}\gamma}| + \vartheta(q^2)\frac{2}{\pi}\int_0^q \mathrm{d}k|\Psi_{q,k}\rangle\langle\Psi_{q,k}|\right]. \tag{42}$$

Inserting this into equation (5) and using equation (4) we obtain a differential equation for the potential

$$\begin{aligned}\frac{\partial V_3(p,\gamma,k,k';\Lambda)}{\partial\Lambda} = &-\frac{M\bar{R}^4}{2\pi^2}\frac{1}{p^2-\Lambda^2}\Bigg[V_3(p,\gamma,k,\mathrm{i}\gamma;\Lambda)|\Psi_{\Lambda,\mathrm{i}\gamma}(\bar{R},\bar{\Omega})|^2 V_3(p,\gamma,\mathrm{i}\gamma,k';\Lambda)\\ &+\frac{2}{\pi}\int_0^\Lambda \mathrm{d}k'' V_3(p,\gamma,k,k'';\Lambda)|\Psi_{\Lambda,k''}(\bar{R},\bar{\Omega})|^2 V_3(p,\gamma,k'',k';\Lambda)\Bigg]\\ &-\frac{M\bar{R}^4}{4\pi}\sum_{n=0}^{\infty}\frac{|\Psi_n(\bar{R},\bar{\Omega})|^2}{p^2+p_n^2}\delta(\Lambda-p_n)V_3(p,\gamma,k,\mathrm{i}p_n/3;\Lambda)\\ &\times V_3(p,\gamma,\mathrm{i}p_n/3,k';\Lambda).\end{aligned} \tag{43}$$

For $R \ll a_2$ all the DWs will be dominated by the s_0 channel and will therefore tend to the forms

$$\begin{aligned}|\Psi_{p,k}(R,\Omega)|^2 &\sim \frac{1}{p}\mathcal{D}_3(p,k,\gamma,p_*)|\xi(\Omega)|^2\frac{\sin^2(s_0\ln(p_*R)-\theta)}{R^4},\\ |\Psi_{p,\mathrm{i}\gamma}(R,\Omega)|^2 &\sim \mathcal{D}_2(p,\gamma,p_*)|\xi(\Omega)|^2\frac{\sin^2(s_0\ln(p_*R)-\theta)}{R^4},\\ |\Psi_n(R,\Omega)|^2 &\sim p_n^2\mathcal{D}_B^{(n)}(\gamma,p_*)|\xi(\Omega)|^2\frac{\sin^2(s_0\ln(p_*R)-\theta)}{R^4}.\end{aligned} \tag{44}$$

The various normalization functions, $\mathcal{D}_3$, $\mathcal{D}_2$ and $\mathcal{D}_B^{(n)}$, are determined by the external boundary conditions on the DWs. They can be found by solving the full Faddeev equations (or equivalents). These functions are essential to the RG discussion, since it is through them that information about the long-distance physics is communicated to short distances.

The forms of the DWs above were chosen to ensure that the normalization functions are dimensionless. They have a common dependence on the hyperangles given by the function

$$\xi(\Omega) = \mathcal{A}_{s_0}\sum_{i=1}^{3}\frac{2}{\sin(2\varphi_i)}\sinh\left(\frac{s_0\pi}{2}-s_0\varphi_i\right), \tag{45}$$

where φ_i is the hyperangle defined in appendix A. Since this dependence can be factored out of the Green function $G_2(p)$ at small R, it is unimportant to the RG discussion. In this context, it is convenient to define a Green function with this common short-distance behaviour divided out,

$$\mathcal{G}_2(p,\gamma,p_*) = \frac{2\pi^2\bar{R}^4}{M}\frac{G_2(p;\bar{R},\bar{\Omega};\bar{R},\bar{\Omega})}{|\xi(\bar{\Omega})|^2\sin^2(s_0\ln(p_*\bar{R})-\theta)}. \tag{46}$$

Using the forms of the DWs given in equation (44) we define our rescaled potential by

$$\hat{V}_3(\hat{p},\hat{\gamma},\hat{k},\hat{k}';\Lambda) = \frac{M}{2\pi^2}|\xi(\bar{\Omega})|^2\sin^2(s_0\ln(p_*\bar{R})-\theta)V_3(p,\gamma,k,k';\Lambda), \tag{47}$$

where, as usual, $\hat{p} = p/\Lambda$, etc. We also require rescaled versions of the normalization functions $\mathcal{D}$, which we define by

$$\begin{aligned} &\hat{\mathcal{D}}_3(\hat{k}, \hat{\gamma}, \Lambda) = \mathcal{D}_3(\Lambda, \Lambda\hat{k}, \Lambda\hat{\gamma}, p_*), \qquad \hat{\mathcal{D}}_2(\hat{\gamma}, \Lambda) = \mathcal{D}_2(\Lambda, \Lambda\hat{\gamma}, p_*), \\ &\hat{\mathcal{D}}_B^{(n)}(\hat{\gamma}, \Lambda) = \mathcal{D}_B^{(n)}(\Lambda\hat{\gamma}, p_*), \end{aligned} \tag{48}$$

and the similarly rescaled Green function,

$$\hat{\mathcal{G}}_2(\hat{\gamma}, \Lambda) = \mathcal{G}_2(\Lambda, \Lambda\hat{\gamma}, p_*). \tag{49}$$

Note that p_* and Λ are the only scales in these dimensionless functions and hence they occur only in the ratio Λ/p_*. To simplify notation, we have suppressed the dependence on p_*.

The rescaled potential $\hat{V}_3$ satisfies the RG equation

$$\begin{aligned} \Lambda\frac{\partial \hat{V}_3}{\partial \Lambda} &= \hat{p}\frac{\partial \hat{V}_3}{\partial \hat{p}} + \hat{\gamma}\frac{\partial \hat{V}_3}{\partial \hat{\gamma}} + \hat{k}\frac{\partial \hat{V}_3}{\partial \hat{k}} + \hat{k}'\frac{\partial \hat{V}_3}{\partial \hat{k}'} \\ &\quad + \frac{1}{1-\hat{p}^2}\Bigg[\hat{V}_3(\hat{p}, \hat{\gamma}, \hat{k}, \mathrm{i}\hat{\gamma}; \Lambda)\hat{\mathcal{D}}_2(\hat{\gamma}, \Lambda)\hat{V}_3(\hat{p}, \hat{\gamma}, \mathrm{i}\hat{\gamma}, \hat{k}'; \Lambda) \\ &\quad + \frac{2}{\pi}\int_0^1 \mathrm{d}\hat{k}''\,\hat{V}_3(\hat{p}, \hat{\gamma}, \hat{k}, \hat{k}''; \Lambda)\hat{\mathcal{D}}_3(\hat{k}'', \hat{\gamma}, \Lambda)\hat{V}_3(\hat{p}, \hat{\gamma}, \hat{k}'', \hat{k}'; \Lambda)\Bigg] \\ &\quad + \frac{\pi}{2}\sum_{n=0}^{\infty}\hat{\mathcal{D}}_B^{(n)}(\hat{\gamma}, \Lambda)\frac{1}{1+\hat{p}^2}\delta(\hat{p}_n(\Lambda) - 1)\hat{V}_3(\hat{p}, \hat{\gamma}, \hat{k}, \mathrm{i}\hat{p}_n/3; \Lambda) \\ &\quad \times \hat{V}_3(\hat{p}, \hat{\gamma}, \mathrm{i}\hat{p}_n/3, \hat{k}'; \Lambda). \end{aligned} \tag{50}$$

The boundary conditions on $\hat{V}_3$ are that it be analytic in $\hat{p}^2$, $\hat{\gamma}$, $\hat{k}^2$ and $\hat{k}'^2$. Again these follow from our requirement that the potential arises from six-point contact terms in the effective Lagrangian.

In the $\hat{\mathcal{D}}$ and the rescaled bound-state momenta $\hat{p}_n(\Lambda)$ the only explicit dependence on Λ occurs as a ratio with the scale p_* introduced, as before, by the self-adjoint extension. All quantities appearing in the RG equation are invariant under the transformation in equation (7) and hence are periodic in p_*. Furthermore, this implies that the equation must be unchanged under

$$\Lambda \to \Lambda\,\mathrm{e}^{-n\pi/s_0}, \qquad n \in \mathbb{Z}. \tag{51}$$

In interpreting this for the bound state momenta $\hat{p}_n(\Lambda)$ we need to be careful. Depending on how we label the states, these momenta may not obey the symmetry individually. However, the complete set must always obey it collectively.

To see how this happens, consider the entire rescaled spectrum, $\hat{p}_n$, at some initial scale $\Lambda = \Lambda_1$. As Λ decreases the rescaled bound state momenta increase, until at $\Lambda = \Lambda_1\,\mathrm{e}^{-\pi/s_0}$ the spectrum has shifted so that $\hat{p}_n(\Lambda) = \hat{p}_{n+1}(\Lambda')$. This is similar to what happened in the case of infinite scattering length, except that a new bound state must have appeared at threshold and moved down to $\hat{p}_0(\Lambda_1)$. This is possible because in following the RG flow we keep the rescaled quantity $\hat{\gamma}$ fixed, not the physical value γ. The spectrum at $\Lambda = \Lambda_1\,\mathrm{e}^{-\pi/s_0}$ is thus identical to that at Λ_1. If we label the bound states so that $\hat{p}_0(\Lambda)$ always refers to the shallowest, then each of the $\hat{p}_n(\Lambda)$ will be invariant under equation (51).

The general RG equation is complicated because it includes couplings between the different three-body channels. As a starting point, we consider a solution which is independent of the two relative momenta, $\hat{k}$ and $\hat{k}'$. This simplifies the equation considerably since it allows

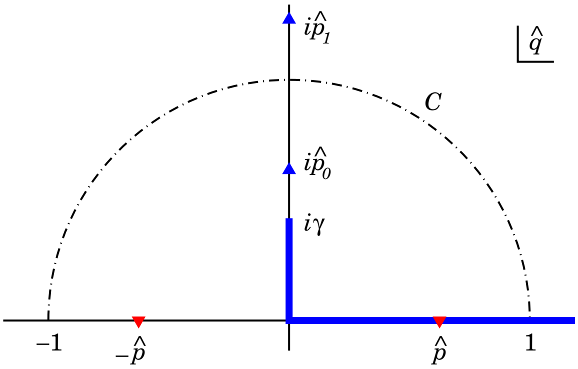

Figure 2. The contour C used to construct the DWRG solution $\hat{J}(\hat{p}, \hat{\gamma}, \Lambda)$. The bound-state poles of the integrand $H(\hat{q})$ occur at $\hat{q} = \mathrm{i}\hat{p}_n$. The branch cut running down the imaginary axis corresponds to the $2+1$ elastic continuum. This is joined by a cut starting at $\hat{q} = 0$ corresponding to the three-body continuum. There are also propagator poles at $\hat{q} = \pm\hat{p}$.

us to divide through by $\hat{V}_3(\hat{p}, \hat{\gamma}; \Lambda)^2$ and obtain a linear equation for $1/\hat{V}_3$,

$$\Lambda \frac{\partial}{\partial \Lambda}\left(\frac{1}{\hat{V}_3}\right) = \hat{p}\frac{\partial}{\partial \hat{p}}\left(\frac{1}{\hat{V}_3}\right) + \hat{\gamma}\frac{\partial}{\partial \hat{\gamma}}\left(\frac{1}{\hat{V}_3}\right) - \frac{1}{1-\hat{p}^2}\hat{P}(\hat{\gamma}, \Lambda) + \frac{\pi}{2}\sum_{n=0}^{\infty}\hat{\mathcal{D}}_B^{(n)}(\hat{\gamma}, \Lambda)\frac{1}{1+\hat{p}^2}\delta(\hat{p}_n(\Lambda) - 1), \tag{52}$$

where we have introduced the function

$$\hat{P}(\hat{\gamma}, \Lambda) = \hat{\mathcal{D}}_2(\hat{\gamma}, \Lambda) + \frac{2}{\pi}\int_0^1 \mathrm{d}\hat{k}''\hat{\mathcal{D}}_3(\hat{k}'', \hat{\gamma}, \Lambda). \tag{53}$$

This function can be thought of as the (rescaled) short-distance part of the projection operator onto continuum states with energy $E = \Lambda^2/M$.

Our strategy for solving equation (52) will mirror that used in the simpler case of infinite scattering length. As before, we can immediately write down a solution to the continuous equation using an equivalent to the basic loop integral, equation (19). Again this does not satisfy the analyticity boundary conditions because of the singular behaviour of the integrand at the endpoint $\hat{q} = 0$. Also, this solution does not contain the discontinuities at the bound states. We therefore need a generalization of equation (20) which allows us to write the solution as a contour integral that avoids $\hat{q} = 0$.

Guided by the results for infinite scattering length, we define an integral along the same contour C as before,

$$\hat{J}(\hat{p}, \hat{\gamma}, \Lambda) = \int_C \mathrm{d}\hat{q}\, H(\hat{q}), \tag{54}$$

where

$$H(\hat{q}) = \frac{\mathrm{i}}{\pi}\frac{\hat{q}}{\hat{p}^2 - \hat{q}^2}\hat{\mathcal{G}}_2(\hat{\gamma}/\hat{q}, \hat{q}\Lambda). \tag{55}$$

The contour C is shown in figure 2 along with the singularity structure of H. This structure follows straightforwardly from the form of $\hat{\mathcal{G}}_2$ and is derived in appendix B.

In appendix B we also show that the function $\hat{P}(\hat{\gamma}, \Lambda)$ in equation (52) can be rewritten in terms of $\hat{\mathcal{G}}_2$ as

$$\hat{P}(\hat{\gamma}, \Lambda) = \frac{\mathrm{i}}{\pi}(\hat{\mathcal{G}}_2(\hat{\gamma}, \Lambda) - \hat{\mathcal{G}}_2(-\hat{\gamma}, -\Lambda)). \tag{56}$$

This result can then be used to show that $\hat{J}(\hat{p}, \hat{\gamma}, \Lambda)$ satisfies the continuous part of equation (52). In addition, the bound state poles in H cross the contour of integration at $\hat{p}_n = 1$ and so produce the necessary discontinuities. Finally, the contour never approaches the branch points at $\hat{q} = 0$ and $\hat{q} = \mathrm{i}\hat{\gamma}$ and so the integral is analytic in the small scales. This allows us to take our nontrivial solution to the DWRG equation to be

$$\hat{V}_3^{(0)}(\hat{p}, \hat{\gamma}, \Lambda) = [\hat{J}(\hat{p}, \hat{\gamma}, \Lambda)]^{-1}. \tag{57}$$

The periodic dependence on Λ of all quantities appearing in the DWRG equation (52) implies that our solution $\hat{V}_3^{(0)}(\hat{p}, \hat{\gamma}, \Lambda)$ must also be periodic. Like the analogous solution for infinite scattering length, it is therefore a limit-cycle of the DWRG.

As a result, it must have an infinite number of discontinuities as $\Lambda \to 0$, which follow a periodic pattern according to this symmetry. At first glance this may seem odd since these discontinuities are associated with the truncation of bound states and there are only a finite number of bound states in the truncated Green function. However, as Λ is lowered for fixed $\hat{\gamma}$, new bound states appear in the rescaled spectrum as discussed above.

Now that we have obtained a limit-cycle solution, the analysis proceeds along very similar lines to that in the previous section. The perturbations in $\hat{p}$ and $\hat{\gamma}$ are simple to derive and, since equation (52) is linear, they give exact solutions to the DWRG equation:

$$\frac{1}{\hat{V}_3(\hat{p}, \hat{\gamma}, \Lambda)} = \frac{1}{\hat{V}_3^{(0)}(\hat{p}, \hat{\gamma}, \Lambda)} + \sum_{n,m=0}^{\infty} C_{2n,m} \Lambda^{2n+m} \hat{p}^{2n} \hat{\gamma}^m. \tag{58}$$

The main features of the power counting are unaffected by the addition of the new scale γ. The LO perturbation is still marginal, and leads to a family of limit-cycles parametrized by $C_{0,0}$. Although it is more difficult to show here, the analysis of the infinite-scattering-length case suggests that the marginal perturbation $C_{0,0}$ and the scale p_* must represent the same three-body physics in the EFT.

The previous power-counting is supplemented by terms proportional to powers of γ. In many systems, such as those of interest in nuclear physics, the two-body scattering lengths and hence γ are fixed. This means that these terms cannot be determined separately but must be absorbed into the coefficients of the energy-dependent perturbations. However, in atomic systems with tunable Feshbach resonances, it may be possible to vary γ and hence to disentangle the energy- and γ-dependent perturbations.

The full set of perturbations also includes ones that depend on the relative momenta. The forms of these have been derived and their eigenvalues show that they occur at the same orders as the corresponding energy-dependent perturbations. The full expressions are very unwieldy and so the interested reader is referred to [18].

Having obtained a solution to the RG equation we can substitute it back into the LS equation to obtain the corresponding DW T-matrix. The algebra again follows that for systems with infinite scattering length. In the present case, the truncated Green function involves an integral around the branch cut corresponding to elastic scattering of a single particle on a bound pair below the three-body threshold. This 2 + 1 continuum cut runs along the imaginary $\hat{q}$-axis from $\mathrm{i}\hat{\gamma}$, as shown in figure 2. Using the form for the three-body force given in equation (58) we find that the DW T-matrix element for 2 + 1 elastic scattering can be written as

$$\frac{2\pi^2}{M} \frac{\mathcal{D}_2(p, \gamma, p_*)\, \mathrm{e}^{2\mathrm{i}\delta_2(p)}}{\langle \Psi^-_{p,\mathrm{i}\gamma} | \tilde{T}_3(p) \big| \Psi^+_{p,\mathrm{i}\gamma} \big\rangle} = -\mathcal{G}_2(p, \gamma, p_*) + \sum_{n,m=0}^{\infty} C_{2n,m} p^{2n} \gamma^m. \tag{59}$$

Using the properties of $\mathcal{G}_2$ in appendix B, one can show that $\tilde{T}_3(p)$ respects unitarity for $p^2 < 0$, as it should. For $p^2 > 0$, the coupling to three-body break-up channels, mean that T-matrix for 2 + 1 scattering does not respect unitarity on its own. The same effective potential could be used to calculate amplitudes for break-up and $3 \to 3$ scattering. However, in practice this is likely to require general momentum-dependent potentials to describe all possible interaction channels.

The DW T-matrix can be related to the elastic-scattering phase shift, δ, by

$$\frac{\mathrm{e}^{2\mathrm{i}\delta_2(p)}}{\langle\Psi^-_{p,i\gamma}|\tilde{T}_3(p)\big|\Psi^+_{p,\mathrm{i}\gamma}\big\rangle} = -\frac{M}{4\pi}[\cot(\delta(p) - \delta_2(p)) - \mathrm{i}], \tag{60}$$

where δ_2 is the phase shift produced by pairwise forces only. This allows us to rewrite equation (59) in the form of a DW effective-range expansion,

$$\frac{\pi}{2}\mathcal{D}_2(p,\gamma,p_*)[\cot(\delta(p) - \delta_2(p)) - \mathrm{i}] - \mathcal{G}_2(p,\gamma,p_*) = -\sum_{n,m=0}^{\infty} C_{2n,m}p^{2n}\gamma^m. \tag{61}$$

Unlike the standard effective-range expansion for three-body systems (see, e.g., [27]), this expression provides an expansion of the residual scattering after the effects of the two-body interaction have been removed. In it, all of the nonanalytic dependences on p and γ have been absorbed into the functions $\mathcal{D}_2(p,\gamma,p_*)$, $\mathcal{G}_2(p,\gamma,p_*)$ and $\delta_2(p)$ to leave a quantity that can be expanded in powers of energy (p^2) and γ. Note that, in contrast to the case of infinite scattering length, the functions here need to be determined numerically. As a result there are no simple analytic expressions for the T- and S-matrices, such as equations (33) and (34) above.

5. Discussion

In this paper we have applied a DW version of the RG to three-body scattering. This is based on cutting off the full solutions in the presence of long-range interactions [15]. In three-body systems, these long-range forces are generated by point-like interactions between pairs of particles. By imposing the cut-off on the DWs we ensure that it affects only the contributions of the short-range three-body forces, and does not alter the two-body physics. Demanding that observables are independent of the cut-off then leads to an RG equation for the renormalized three-body interactions.

As shown by Efimov [27], a three-body system with contact interactions can be described by an ISP in the region where they are all close together. Hence the RG in presence of this rather singular potential controls the behaviour of the three-body forces and determines their power counting.

For systems of three bosons or two neutrons and a proton with total spin 1/2, the resulting ISP is attractive. This leads to quite different behaviour compared with previous examples studied with this type of RG [13, 15]. The wavefunctions show oscillatory behaviour at short distances and a self-adjoint extension is needed to fix their phase and hence give well-defined eigenfunctions [29, 33–35]. These oscillatory functions control the RG flow of the short-distance potential, which tends to a limit cycle [37] rather than a nontrivial fixed point.

These results are general for an attractive ISP in any number of dimensions. The lack of a ground state for such a potential means that it has deeply bound states that lie outside the domain of validity of our effective theory. This means that the cut-off should truncate not only high-energy scattering states but also these deeply bound states. This leads to a three-body force that has periodic discontinuities, but is single valued.

The full three-body interaction can be built out of perturbations around this single-valued limit-cycle solution. The lowest-order term is marginal, in the sense that its RG flow is only logarithmic in the cut-off. From its effect on observables, we have seen that this term is equivalent to changing the parameter in the self-adjoint extension or, in other words, the phase of the short-distance wavefunctions.

To express the resulting power counting in a form which matches that usually used [8, 9], we can assign an order $d = \mu - 1$ to a term in the rescaled potential proportional to Λ^{μ}. Then a term proportional to p^{2n} (or E^n) appears at order $d = 2n - 1$. The coefficients in our potential have direct, and relatively simple, connection to scattering observables.

The same behaviour is found in an attractive three-body system with finite two-body scattering length, a_2, since the wavefunctions tend to the Efimov form at short distances ($R \ll a_2$). Hence the RG flow has a very similar pattern, tending to a limit cycle. In this case, the power counting should be extended to include terms involving powers of the low-energy scale $\gamma = 1/a_2$. A term proportional to$p^{2n}\gamma^m$ is of order $d = 2n + m - 1$. For the energy-dependent perturbations, this counting agrees with that found by Bedaque *et al* [23] from the STM equation. Because of the clean separation of the short- and long-range physics in the DWRG approach we are able to state the counting in an algebraically much simpler way.

We are also able to extend the counting to terms involving powers of γ. This may not be relevant to applications in nuclear physics, where the two-body scattering lengths are fixed, but in atomic systems it may be possible to use Feshbach resonances to vary the scattering length, and hence to disentangle the two types of perturbation.

The leading, marginal three-body force, or equivalently the choice of self-adjoint extension, is of interest since it shows that one piece of three-body physics is needed to determine low-energy three-body observables. It has long been known that the 'Phillips line,' a correlation between the $J = \frac{1}{2}\, nd$ scattering lengths and triton binding energies predicted by model nucleon–nucleon potentials [5] can be explained by three-body interactions [6]. EFT approaches, such this work and [20, 23], provide a systematic way to do this, which can be extended to include higher order interactions and couplings to external currents.

At leading order ($d = -1$), three-body observables are determined by the two-body scattering length and the marginal three-body force. The first correction to this arises from the two-body effective range, which appears at order $d = 0$. Including this has been shown to give good agreement with $J = \frac{1}{2}\, nd$ observables, in calculations using the STM equation [23] and an equivalent equation in coordinate-space [18]. The first energy-dependent three-body force appears only at next-to-next-to-leading order, $d = 1$ in this counting.

It should be possible to extend our approach to three-body scattering above the break-up threshold. However, this will introduce couplings between various channels and so will involve momentum-dependent perturbations in the effective potential. This is likely to require similar analyses to those which have been applied to simpler coupled-channel systems, such as that considered in [44].

Acknowledgments

This work was supported by the EPSRC. MCB acknowledges the Institute for Nuclear Theory, Seattle for hospitality during the programme on Effective Field Theories and Effective Interactions where the seeds of these ideas were sown, and T Cohen, S Coon, J McGovern and R Perry for useful discussions. He is also grateful for discussions with A C Phillips, a much missed colleague and friend.

Appendix A. Efimov wavefunctions

We summarize here the essential elements of Efimov's approach to the three-body problem [27], since this provides the wavefunctions we need to elucidate the scaling of the three-body interactions.

We consider the case of s-wave scattering. In general, the wavefunctions $|\Psi_{p,k}\rangle$ must be found using the Faddeev equations or equivalent (see, e.g., [42]). In the Faddeev formalism, the wavefunction is broken into three components according to the pair of particles that interacted last. The component in which particles 2 and 3 interacted last is denoted by $\psi^{(1)}_{p,k}(r_{23}, r_1)$, where r_{23} and r_1 are the usual Jacobian coordinates for three bodies with equal masses.

For particles interacting via pairwise contact interactions, Efimov observed that $\psi^{(i)}_{p,k}(r_{jk}, r_i)$ satisfies a free two-dimensional Schrödinger equation, subject to a boundary condition relating the wavefunction at the points $r_{jk} = 0$ and $r_i = r_{jk}/2$ where pairs of particles interact. This boundary condition takes the form

$$\left[\frac{\partial\psi^{(i)}_{p,k}(R,\varphi_i)}{\partial\varphi_i}\right]_{\varphi_i=0} + \frac{8\lambda}{\sqrt{3}}\psi^{(i)}_{p,k}(R,\pi/3) = \frac{R}{a_2}\psi^{(i)}_{p,k}(R,0), \tag{A.1}$$

where λ is a factor related to the wavefunction symmetry, and R and φ_i are the hyperradius and hyperangle respectively, and are defined by

$$R = \sqrt{\frac{4}{3}r_i^2 + r_{jk}^2}, \qquad \varphi_i = \arctan\frac{\sqrt{3}r_{jk}}{2r_i}. \tag{A.2}$$

For three bosons, the full wavefunction can be expressed as

$$\Psi_{p,k}(R,\Omega) = \sum_{i=1}^{3}\frac{2}{R^2\sin(2\varphi_i)}\psi^{(i)}_{p,k}(R,\varphi_i), \tag{A.3}$$

where Ω represents five general hyperspherical coordinates that complement R. (The other coordinates could be φ_i and two angles each for $\mathbf{r}_i$ and $\mathbf{r}_{jk}$, but their exact specification will not be needed here.)

We shall assume that the DWs are normalized by

$$\int_0^\infty R^5\,\mathrm{d}R\int \mathrm{d}\Omega\Psi_{p,k}(R,\Omega)\Psi_{p',k'}(R,\Omega) = \frac{\pi^2}{4}\delta(p^2-p'^2)\delta(k-k'), \tag{A.4}$$

$$\int_0^\infty R^5\,\mathrm{d}R\int \mathrm{d}\Omega\Psi_{p,\mathrm{i}\gamma}(R,\Omega)\Psi_{p',\mathrm{i}\gamma}(R,\Omega) = \frac{\pi}{2}\delta(p^2-p'^2), \tag{A.5}$$

$$\int_0^\infty R^5\,\mathrm{d}R\int \mathrm{d}\Omega\Psi_n(R,\Omega)\Psi_n(R,\Omega) = 1. \tag{A.6}$$

Because the boundary condition couples R and φ_i, the equations are, in general, extremely difficult to solve (see, e.g., [45]). However, when $R \ll a_2$ the boundary condition becomes separable. In this limit we may label the states in terms of the centre-of-mass energy and the hyperangular momentum, s. We can write the solutions in the form

$$\psi^{(i)}_{p,s}(R,\varphi_i) = \mathcal{A}_s\sin\left(\frac{s\pi}{2} - s\varphi_i\right)u_s(p,R). \tag{A.7}$$

This satisfies the angular Schrödinger equation subject to the boundary condition of vanishing amplitude at $\varphi_i = \pi/2$ ($r_i = 0$). The boundary condition, equation (A.1), now results in a transcendental equation for s,

$$s\cos\frac{s\pi}{2} = \frac{8\lambda}{\sqrt{3}}\sin\frac{s\pi}{6}. \tag{A.8}$$

This equation was also obtained much earlier by Danilov [43] using the momentum–space equation derived by Skorniakov and Ter–Martirosian [24].

The resulting radial Schrödinger equation has the form

$$-\frac{\mathrm{d}^2 u_s(p,R)}{\mathrm{d}R^2} - \frac{1}{R}\frac{\mathrm{d}u_s(p,R)}{\mathrm{d}R} + \frac{s^2}{R^2}u_s(p,R) = p^2 u_s(p,R), \tag{A.9}$$

and so contains an ISP whose strength is determined by s, and hence depends on the symmetry parameter. This Hamiltonian is scale free, which should not be surprising given that it corresponds to the nontrivial fixed point of the two-body system. For three bosons or a neutron and deuteron with spin-1/2, the symmetry parameter is $\lambda = 1$, while for three nucleons with spin-3/2 it is $\lambda = -1/2$. In the latter case, equation (A.8) has real solutions and inverse-square potential provides a repulsive 'centrifugal barrier'. The power counting for short-range interactions in the presence of this potential can be derived using the RG method of [15].

The case with $\lambda = 1$ is more interesting, since the lowest solutions to equation (A.8) are imaginary,

$$s = \pm \mathrm{i}s_0 = \pm \mathrm{i}1.006\ldots, \tag{A.10}$$

and so they correspond to an attractive ISP. For small R the solutions to the corresponding radial equation have the forms

$$u_{s_0}(p,R) \propto R^{\pm \mathrm{i}s_0}. \tag{A.11}$$

These are both equally (ir)regular as $R \to 0$. They can also describe flux disappearing or being created at the origin, corresponding to the classical 'fall into the centre' which is possible for this potential [32]. To obtain well-defined wavefunctions which respect flux conservation, we need to impose a boundary condition on the wavefunctions for $R \to 0$, requiring equal admixtures of the two complex solutions and fixing their relative phase. Mathematically, this is known as choosing a self-adjoint extension of the Hamiltonian in equation (A.9), as discussed by Bawin and Coon [29] and references therein.

The full solutions to the Schrödinger equation with an attractive ISP can be written in terms of Bessel functions of imaginary order,

$$u_{s_0}(p,R) = \sqrt{\frac{p\pi}{2}}\frac{1}{2\mathrm{i}|\sin(\eta(p)+\mathrm{i}\pi s_0/2)|}\left[\mathrm{e}^{\mathrm{i}\eta(p)}J_{is_0}(pR) - \mathrm{e}^{-\mathrm{i}\eta(p)}J_{-\mathrm{i}s_0}(pR)\right], \tag{A.12}$$

where $p = \sqrt{ME}$ and the normalization has been fixed by requiring that $u_{s_0}(p,R) \sim \sin(pR+\delta)/\sqrt{R}$ for large R. For small R, these wavefunctions have the form

$$u_{s_0}(p,R) \sim \sqrt{\frac{p\sinh(\pi s_0)}{s_0[\cosh(\pi s_0) - \cos(2\eta(p))]}}\sin(s_0\ln(pR) + \eta(p) - \theta), \tag{A.13}$$

where

$$\theta = \arg\Gamma(1+\mathrm{i}s_0) + s_0\ln 2. \tag{A.14}$$

Demanding that these waves tend to a common, energy-independent form as $R \to 0$ implies that $\eta(p)$ must be of the form

$$\eta(p) = -s_0\ln(p/p_*), \tag{A.15}$$

where p_* is a parameter which fixes the phase of the sine-log oscillations (or equivalently specifies the self-adjoint extension).

The physical meaning of p_* can be seen by noting that the S-matrix for this system is

$$\mathrm{e}^{2\mathrm{i}\delta_2(p)} = \mathrm{i}\frac{\sin(\eta(p) - \mathrm{i}\pi s_0/2)}{\sin(\eta(p) + \mathrm{i}\pi s_0/2)}. \tag{A.16}$$

This has a pole at $p = \mathrm{i}p_*$, implying that $E = -p_*^2/M$ corresponds to a bound state. In fact, as shown by Efimov, there is an infinite tower of bound states with energies $E = -p_n^2\big/M$ where

$$p_n = p_* \,\mathrm{e}^{n\pi/s_0}. \tag{A.17}$$

The bound states accumulate at zero energy and extend downwards in a geometric pattern with no ground state. The wavefunctions of the bound states are

$$u_{s_0}^{(n)}(R) = \sqrt{\frac{2\sinh(\pi s_0)}{\pi s_0}}\, p_n K_{\mathrm{i}s_0}(p_n R), \tag{A.18}$$

where $K_m(x)$ denotes a modified Bessel function of the third kind. Near the origin these have the form

$$u_{s_0}^{(n)}(R) \sim \frac{\sqrt{2}}{s_0} p_n (-1)^{n+1} \sin(s_0 \ln p_* R - \theta). \tag{A.19}$$

The shallow states are referred to as Efimov states, although the lack of a ground state had been noted much earlier by Thomas [46].

Appendix B. Rescaled projection operator

The short-distance Green function defined in equation (46) has the spectral representation

$$\begin{aligned}\mathcal{G}_2(p,\gamma,p_*) = {}& \int_{-\gamma^2}^{\infty} \mathrm{d}(q^2) \frac{1}{p^2 - q^2 + \mathrm{i}\epsilon}\left[\mathcal{D}_2(q,\gamma,p_*) + \vartheta(q^2)\frac{2}{\pi}\int_0^q \frac{\mathrm{d}k}{q}\mathcal{D}_3(q,k,\gamma,p_*)\right] \\ & + \frac{\pi}{2}\sum_{n=0}^{\infty}\mathcal{D}_B^{(n)}(\gamma,p_*)\frac{p_n^2}{p^2+p_n^2}.\end{aligned} \tag{B.1}$$

Using the result,

$$\lim_{\epsilon\to 0}\int_0^{\infty}\frac{f(x)\mathrm{d}x}{x_0 - x \pm \mathrm{i}\epsilon} = \mathcal{P}\int_0^{\infty}\frac{f(x)\mathrm{d}x}{x_0 - x} \mp \mathrm{i}\pi f(x_0), \tag{B.2}$$

where $\mathcal{P}$ denotes the principal value, we find that, for real p,

$$\frac{\mathrm{i}}{\pi}[\mathcal{G}_2(p,\gamma,p_*) - \mathcal{G}_2(-p,\gamma,p_*)] = \mathcal{D}_2(p,\gamma,p_*) + \frac{2}{\pi}\int_0^p \frac{\mathrm{d}k}{p}\mathcal{D}_3(p,k,\gamma,p_*). \tag{B.3}$$

In this equation $\mathcal{G}_2(-p,\gamma,p_*)$ is found by analytically continuing p through the upper half of the complex p plane to $-p$. Since, by its definition, $\mathcal{G}_2(p,\gamma,p_*)$ is real for pure imaginary p, its value at negative real p is the complex conjugate of that at positive real p and hence $\mathcal{G}_2(-p,\gamma,p_*)$ corresponds to a $-\mathrm{i}\epsilon$ prescription at the propagator pole.

By rescaling equation (B.3), we obtain an expression for the rescaled projection operator defined in equation (53):

$$\hat{P}(\hat{\gamma},\Lambda) = \frac{\mathrm{i}}{\pi}(\hat{\mathcal{G}}_2(\hat{\gamma},\Lambda) - \hat{\mathcal{G}}_2(-\hat{\gamma},-\Lambda)), \tag{B.4}$$

where $\hat{\mathcal{G}}_2$ is the rescaled Green function defined in equation (49).

That rescaled Green function has the representation

$$\begin{aligned}\hat{\mathcal{G}}_2(\hat{\gamma},\Lambda) = {}& \int_{-\hat{\gamma}^2}^{\infty}\mathrm{d}(\hat{q}^2)\frac{1}{1-\hat{q}^2+\mathrm{i}\epsilon}\left[\hat{\mathcal{D}}_2\left(\frac{\hat{\gamma}}{\hat{q}},\hat{q}\Lambda\right) + \vartheta(\hat{q}^2)\frac{2}{\pi}\int_0^1 \mathrm{d}\hat{k}\hat{\mathcal{D}}_3\left(\frac{\hat{k}}{\hat{q}},\frac{\hat{\gamma}}{\hat{q}},\hat{q}\Lambda\right)\right] \\ & + \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\hat{\mathcal{D}}_B^{(n)}(\hat{\gamma},\Lambda)\hat{p}_n(\Lambda)^2}{1+\hat{p}_n(\Lambda)^2}.\end{aligned} \tag{B.5}$$

The analytic properties of this function are needed since it forms part of the integrand $H(\hat{q})$ of equation (55), which is used to construct the RG limit-cycle solution. From its spectral representation, we see that $\hat{\mathcal{G}}_2(\hat{\gamma}, \Lambda)$ has poles at the bound state momenta, $\hat{q} = \mathrm{i}\hat{p}_n(\Lambda)$. The integral term results in a branch cut down the imaginary axis from $\hat{q} = \mathrm{i}\hat{\gamma}$ and then along the positive real axis.

References

[1] Beane S R, Bedaque P F, Haxton W C, Phillips D R and Savage M J 2001 *At the Frontier of Particle Physics: Handbook of QCD* vol 1 ed M Shifman (Singapore: World Scientific) p 133 (*Preprint* nucl-th/0008064)
[2] Bedaque P F and van Kolck U 2002 *Ann. Rev. Nucl. Part. Sci.* **52** 339 (*Preprint* nucl-th/0203055)
[3] Bethe H A 1949 *Phys. Rev.* **76** 38
[4] Blatt J M and Jackson J D 1949 *Phys. Rev.* **76** 18
[5] Phillips A C 1966 *Phys. Rev.* A **142** 984
Phillips A C 1968 *Nucl. Phys.* **107** 209
[6] Efimov V and Tkachenko E G 1985 *Phys. Lett.* B **157** 108
[7] Brayshaw D D 1975 *Phys. Rev.* D **8** 952
Brayshaw D D 1976 *Phys. Rev.* C **13** 1024
[8] Weinberg S 1979 *Physica* A **96** 327
[9] Weinberg S 1990 *Phys. Lett.* B **251** 288
Weinberg S 1991 *Nucl. Phys.* B **363** 3
[10] van Kolck U 1999 *Nucl. Phys.* A **645** 273 (*Preprint* nucl-th/9808007)
[11] Kaplan D B, Savage M J and Wise M B 1998 *Phys. Lett.* B **424** 390 (*Preprint* nucl-th/9801034)
Kaplan D B, Savage M J and Wise M B 1998 *Nucl. Phys.* B **534** 329 (*Preprint* nucl-th/9802075)
[12] Wilson K G and Kogut J G 1974 *Phys. Rep.* B **12** 75
Polchinski J 1984 *Nucl. Phys.* B **231** 269
[13] Birse M C, McGovern J A and Richardson K G 1999 *Phys. Lett.* B **464** 169 (*Preprint* hep-ph/9807302)
[14] Gegelia J 1999 *J. Phys. G: Nucl. Part. Phys.* **25** 1681 (*Preprint* nucl-th/9805008)
[15] Barford T and Birse M C 2003 *Phys. Rev.* C **67** 064006 (*Preprint* hep-ph/0206146)
[16] van Haeringen H and Kok L P 1982 *Phys. Rev.* A **26** 1218
[17] Barford T and Birse M C 2003 *Few Body Syst. Suppl.* **14** 123 (*Preprint* nucl-th/0210084)
[18] Barford T 2004 *PhD Thesis* University of Manchester (*Preprint* nucl-th/0404072)
[19] Bedaque P F and van Kolck U 1998 *Phys. Lett.* B **428** 221 (*Preprint* nucl-th/9710073)
Bedaque P F, Hammer H-W and van Kolck U 1998 *Phys. Rev.* C **58** 641 (*Preprint* nucl-th/9802057)
[20] Bedaque P F, Hammer H-W and van Kolck U 1999 *Phys. Rev. Lett.* A **82** 463 (*Preprint* nucl-th/9809025)
Bedaque P F, Hammer H-W and van Kolck U 1999 *Nucl. Phys.* **646** 444 (*Preprint* nucl-th/9811046)
Bedaque P F, Hammer H-W and van Kolck U 2000 *Nucl. Phys.* A **676** 357 (*Preprint* nucl-th/9906032)
[21] Bedaque P F, Braaten E and Hammer H-W 2000 *Phys. Rev. Lett.* **85** 908 (*Preprint* cond-mat/0002365)
[22] Hammer H-W and Mehen T 2001 *Nucl. Phys.* A **690** 535 (*Preprint* nucl-th/0011024)
[23] Bedaque P F, Greißhammer H W, Hammer H-W and Rupak G 2003 *Nucl. Phys.* A **714** 589 (*Preprint* nucl-th/0207034)
[24] Skorniakov G V and Ter-Martirosian K A 1957 *Sov. Phys.—JETP* **4** 648
[25] Afnan I R and Phillips D R 2004 *Phys. Rev.* C **69** 034010 (*Preprint* nucl-th/0312021)
[26] Mohr R F 2003 *PhD Thesis* Ohio State University (*Preprint* nucl-th/0306086)
[27] Efimov V N 1971 *Sov. J. Nucl. Phys.* **12** 589
Efimov V N 1979 *Sov. J. Nucl. Phys.* **29** 546
[28] Phillips A C 1977 *Rep. Prog. Phys.* **40** 905
[29] Bawin M and Coon S A 2003 *Phys. Rev.* A **67** 042712 (*Preprint* quant-ph/0302199)
[30] Camblong H E and Ordóñez C R 2003 *Preprint* hep-th/0305035
[31] Braaten E and Phillips D 2004 *Preprint* hep-th/0403168
[32] Newton R G 1982 *Scattering Theory of Waves and Particles* (New York: Springer)
[33] Case K M 1950 *Phys. Rev.* **80** 797
[34] Meetz K Il 1964 *Nuovo Cimento* **34** 690
[35] Perelomov A M and Popov V S 1970 *Theor. Math. Phys.* **4** 664
[36] Beane S R, Bedaque P F, Childress L, Kryjevski A, McGuire J and van Kolck U 2001 *Phys. Rev.* A **64** 042103 (*Preprint* quant-ph/0010073)

[37] Wilson K G 2000 *Talk Presented at the Institute for Nuclear Theory, Seattle* unpublished
R J Perry 2000, Private communication
[38] Gazek S D and Wilson K G 2002 *Phys. Rev. Lett.* **89** 230401 (*Preprint* hep-th/0203088)
Gazek S D and Wilson K G 2004 *Phys. Rev.* B **69** 094304 (*Preprint* cond-mat/0303297)
[39] LeClair A, Roman J M and Sierra G 2004 *Phys. Rev.* B **69** 20505 (*Preprint* cond-mat/0211338)
LeClair A, Roman J M and Sierra G 2004 *Nucl. Phys.* B **700** 407 (*Preprint* hep-th/0312141)
[40] Mueller E J and Ho T-L 2004 *Preprint* cond-mat/0403283
[41] Braaten E and Hammer H-W 2003 *Phys. Rev. Lett.* **91** 102002 (*Preprint* nucl-th/0303038)
[42] Thomas A W (ed) 1977 *Modern Three-Hadron Physics* (New York: Springer)
[43] Danilov G S 1961 *Sov. Phys.—JETP* **13** 349
[44] Cohen T D, Gelman B A and van Kolck U 2004 *Phys. Lett.* B **588** 57 (*Preprint* nucl-th/0402054)
[45] Fedorov D V and Jensen A S 1993 *Phys. Rev. Lett.* **71** 4103 (*Preprint* quant-ph/0106039)
Fedorov D V and Jensen A S 2001 *J. Phys. A: Math. Gen.* **34** 6003
Fedorov D V and Jensen A S 2002 *Nucl. Phys.* A **697** 783 (*Preprint* nucl-th/0107027)
[46] Thomas L H 1935 *Phys. Rev.* **47** 903